# Technical Notes

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## Marching Control Volume Finite-Element Calculation for Developing Entrance Flow

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### Introduction

THE analysis of entrance flow in constant-area ducts and pipes is a classical problem in fluid mechanics and has received much attention in the past. This attention includes numerical solution of the Navier-Stokes equations for entrance flow development.[1,2] The Navier-Stokes equations typically are solved with either a finite-difference or a finite-element method. However, use of these numerical analyses presents a problem in that the streamwise extent of the computational domain must be selected without a priori knowledge of the fully developed entrance length $L_f$. The solution is thus constrained.

To overcome the entrance-length problem, a marching control volume approach has been developed that does not require the initial knowledge of $L_f$. The formulation has an additional advantage of avoiding the usual iterative procedure for satisfying the vorticity boundary conditions along the solid wall; instead, a direct approach is used to introduce these boundary conditions. Our method apparently exhibits no convergence problems for laminar flow at Reynolds numbers up to at least 2000.

### Marching Control Volume Algorithm

In this formulation, the dimensionless Navier-Stokes equations for steady, two-dimensional incompressible flow, expressed in terms of the stream function and vorticity, are

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=-\omega \tag{1}$$

$$\frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y}=\frac{1}{Re}\left(\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2}\right) \tag{2}$$

where $\psi$ is the dimensionless stream function and $\omega$ is the vorticity. The domain (Fig. 1) is a channel of constant height; the height is chosen as the unit of length. The velocity profile at the duct inlet is uniformly 1. $Re$ denotes the Reynolds number based on duct height and inlet velocity.

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The Galerkin-weighted residual method is employed to solve Eqs. (1) and (2) approximately. Thus,

$$\int\int_D W\left[\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)+\omega\right]\mathrm{d}x\mathrm{d}y=0 \tag{3}$$

$$\int\int_D W\left[\frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y}-\frac{1}{Re}\left(\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2}\right)\right]\mathrm{d}x\mathrm{d}y=0 \tag{4}$$

where $W$ is the weighting function.

Using Green's formula, Eqs. (3) and (4) reduce to

$$\int\int_D\left(\frac{\partial W}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial W}{\partial y}\frac{\partial\psi}{\partial y}-W\omega\right)\mathrm{d}x\mathrm{d}y-\int_\Gamma W\frac{\partial\psi}{\partial n}\,\mathrm{d}s=0 \tag{5}$$

$$\int\int_D\left[\frac{1}{Re}\left(\frac{\partial W}{\partial x}\frac{\partial\omega}{\partial x}+\frac{\partial W}{\partial y}\frac{\partial\omega}{\partial y}\right)+W\left(\frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y}\right)\right]\mathrm{d}x\mathrm{d}y-\frac{1}{Re}\int_\Gamma W\frac{\partial\omega}{\partial n}\,\mathrm{d}s=0 \tag{6}$$

The derivatives in the normal direction on the downstream boundary vanish; therefore, no downstream boundary integrals exist. Nevertheless, unless the flow region is confined downstream with a preselected, fully developed flow condition, the derivatives in the normal direction will not vanish and the boundary integrals must be performed. The contribution of the boundary integrals is added to the local stiffness matrices of the downstream boundary elements. The unknown stream function and vorticity values along this downstream boundary are then determined by the finite-element matrix system.

To begin, a first control volume is selected. The boundary conditions are $\psi(0,y)=y$, $\omega(0,y)=0$; $\psi(x,0)=0$, $\omega(x,0)=0$; and $\psi(x,0.5)=c$, $\partial\psi/\partial y(x,0.5)=0$; where $c$ is a constant. After solving for the stream function and vorticity, the calculation domain marches downstream to a second control volume. The solutions on the downstream boundary for the previous control volume are used as inlet Dirichlet boundary conditions for the second control volume. Marching ceases when a recognized parabolic profile characteristic of fully developed flow is encountered. The distance from the inlet of the duct to establishment of the parabolic profile is considered to be the fully developed entrance length $L_f$.

### Finite-Element Formulation

The mesh division of the control volume is given in Fig. 1. The downstream length of the control volume is four length units. For simplicity, linear triangular elements are employed

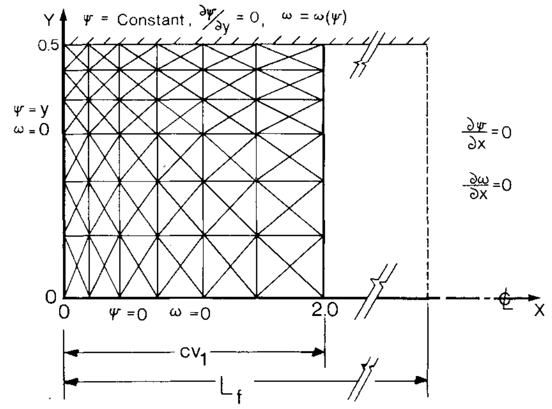


**Fig. 1 Finite-element grid and boundary conditions.**

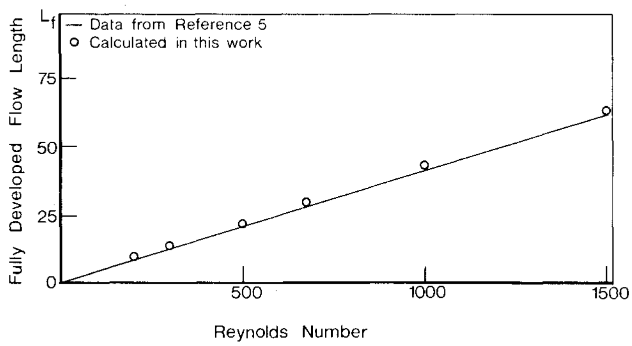


**Fig. 2 Fully developed flow length vs Reynolds number.**

in the finite-element formulation. The whole flow region is divided into 144 elements with 85 nodes.

The weighting function within an element is chosen as the element shape function and a linear variation for both stream function and vorticity is assumed. In the calculation, the velocity components are constant within each element because the stream function-vorticity formulation with the linear triangular elements has been adopted. These velocity component values are considered to be at the geometrical center of the element.

On the elemental basis, Eqs. (5) and (6) can be written as

$$\sum\left[\int\int_{De}\left(\frac{\partial N_i}{\partial x}\frac{\partial\psi}{\partial}+\frac{\partial N_i}{\partial y}\frac{\partial\psi}{\partial y}-N_i\omega\right)\mathrm{d}x\mathrm{d}y - \int_{\Gamma e}N_i\frac{\partial\psi}{\partial n}\mathrm{d}s\right]=0 \tag{7}$$

$$\sum\left[\int\int_{De}\left[\frac{1}{Re}\left(\frac{\partial N_i}{\partial x}\frac{\partial\omega}{\partial x}+\frac{\partial N_i}{\partial y}\frac{\partial\omega}{\partial y}\right)+N_i\left(\frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y}\right)\right]\mathrm{d}x\mathrm{d}y - \frac{1}{Re}\int_{\Gamma_e}N_i\frac{\partial\omega}{\partial n}\mathrm{d}s\right]=0 \tag{8}$$

For elements other than those on the downstream boundary, the local stiffness matrix is written as

$$K_{ij1}=\int\int_{De}\left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}+\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right)\mathrm{d}x\mathrm{d}y$$

$$K_{ij2}=-\int\int_{De}N_iN_j\mathrm{d}x\mathrm{d}y$$

$$K_{ij3}=0$$

$$K_{ij4}=\int\int_{De}\left[\frac{1}{Re}\left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}+\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right)+N_i\left(\frac{\partial\psi}{\partial y}\frac{\partial N_i}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial N_j}{\partial y}\right)\right]\mathrm{d}x\mathrm{d}y \tag{9}$$

The global matrix system constructed on the basis of these local stiffness matrices is

$$[K]=\begin{vmatrix}K_1 & K_2\\ K_3 & K_4\end{vmatrix} \tag{10}$$

with the field variable vector

$$[P]^T=(\psi_1,\psi_2,\ldots,\psi_N;\ \omega_1,\omega_2,\ldots,\omega_N) \tag{11}$$

The derivatives of stream function appearing in $K_{ij4}$ are known from the previous iteration and are updated at the subsequent iteration.

For downstream elements, the boundary integrals have to be carried out because they do not vanish along the downstream boundary. Thus, the auxiliary local stiffness matrix is

$$K_{ij11}=-\int_1^2 N_i\frac{\partial N_j}{\partial x}\mathrm{d}y \tag{12}$$

$$K_{ij44}=-\frac{1}{Re}\int_1^2 N_i\frac{\partial N_i}{\partial x}\mathrm{d}y \tag{13}$$

for $i=1,2$. The discretized equations are solved simultaneously with a method similar to that in Refs. 3 and 4.

The boundary conditions of vorticity along the solid wall are calculated by a local Taylor series expansion[5] after the no-slip condition is applied on the solid wall. The first-order approximation is

$$\omega_i-\frac{2}{dn^2}(\psi_i-\psi_{i+1})=0 \tag{14}$$

where $dn$ is the distance from the solid boundary to the nearest node within the flow, subscript $i$ denotes nodes on the wall, and $i+1$ denotes nodes adjacent and normal to the wall.

Next, the left-hand side of Eq. (14) replaces the $N+i$th line of the global matrix in accordance with the positions of $\psi_i$, $\psi_{i+1}$, and $\omega_i$ in the variable vector. The zero appears in the right-hand side. The procedure is straightforward to implement in a computer program.

## Numerical Results

The developing entrance flows for Reynolds numbers 200, 300, 500, 700, 1000, and 1500 have been calculated. The fully developed entrance lengths are determined when the calculated flow profile is parabolic to less than 1% of the peak velocity. As an example, the fully developed profile is formed at location $x=30.6$, for Reynolds number $=700$. The calculated developing entrance length increases with the increase of Reynolds number (Fig. 2). To show the reliability of the finite-element solutions, an analytical expression for developing entrance flow length from Ref. 6 is employed. The differences between our results and those calculated according to Ref. 5 are less than 5% of our results.

The calculation results also show that the stream function-vorticity finite-element formulation is not degraded by the inclusion of a direct approach for satisfying the updating vorticity boundary conditions along the solid wall.

The program convergence depends on the acceptable error. In the case of 144 elements and 85 nodes, for a Reynolds number of 20, the program coded in FORTRAN converges to within an error of 1.0 ($10^{-10}$) after 7 iterations. For a Reynolds number of 200, 10 iterations are required; and for a Reynolds number of 1500, 15 iterations are needed. We note here that even when the Reynolds number is large, the equations are convectively dominated. Convergence is considered to be reached relatively quickly compared to other formulations.[5,7]

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